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Equations of motion for the "Pendulum" and "Augmented-Reality Pendulum" sketches

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1 Introduction

This document describes the dynamics of a simple pendulum and a double - composite - pendulum systems which are presented in the "Pendulum" Processing sketch, available at `http://www.gwoptics.org/processing/pendulum/`. The same set of equations is also used in the sketch "Augmented-Reality Pendulum", available at `http://www.gwoptics.org/processing/AR_pendulum/`. Screenshots of the two sketches are shown in Fig. 1 and Fig. 2.

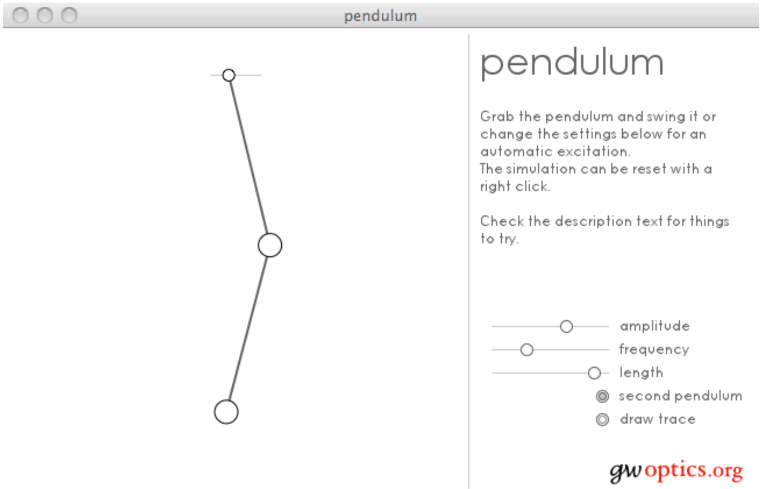

Figure 1: A screenshot of the "Pendulum" sketch.

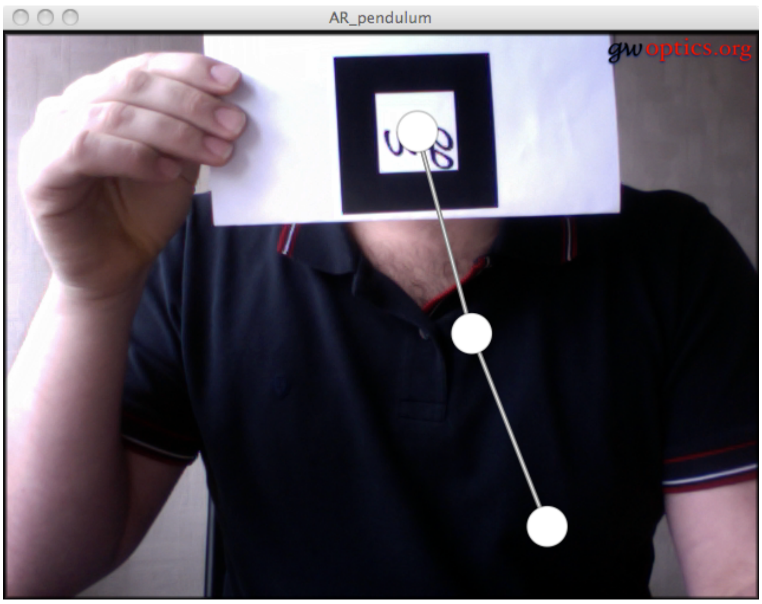

Figure 2: A screenshot of the "Augmented-Reality Pendulum" sketch.

2 Simple Pendulum

2.1 Constants and definitions

A cartoon of the simple pendulum system is shown in Fig. 3-left. The chosen coordinate system is also drawn, with the horizontal axis x directed rightwards and the vertical y axis downwards. The pendulum suspension point has coordinates (x_0, y_0), and the suspended mass m_1 has (x_1, y_1). The other physical parameters relevant for the system are:

- l_1 initial length of pendulum wire at rest (which is defined by the user in the "Pendulum" sketch)
- $d_1 = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ actual length of the suspension wire #1 while suspended
- pendulum's mass m_1
- residual gas damping coefficient β_1
- suspension wire's elastic constants k_1
- suspension's wire anelasticity coefficient δk_1
- θ_1 is the angle that pendulum #1 forms with the vertical axis y, for which we can write $\sin\theta_1 = \frac{(x_1 - x_0)}{d_1}$ and $\cos\theta_1 = \frac{(y_1 - y_0)}{d_1}$.

In the "Pendulum" sketch the pendulum suspension point can be made to oscillate horizontally with user-defined amplitude A and frequency f. In this case the pendulum's suspension horizontal coordinate x_0 can be written as $x_0\,(t) = x_0 + A\sin(2\pi f t)$.

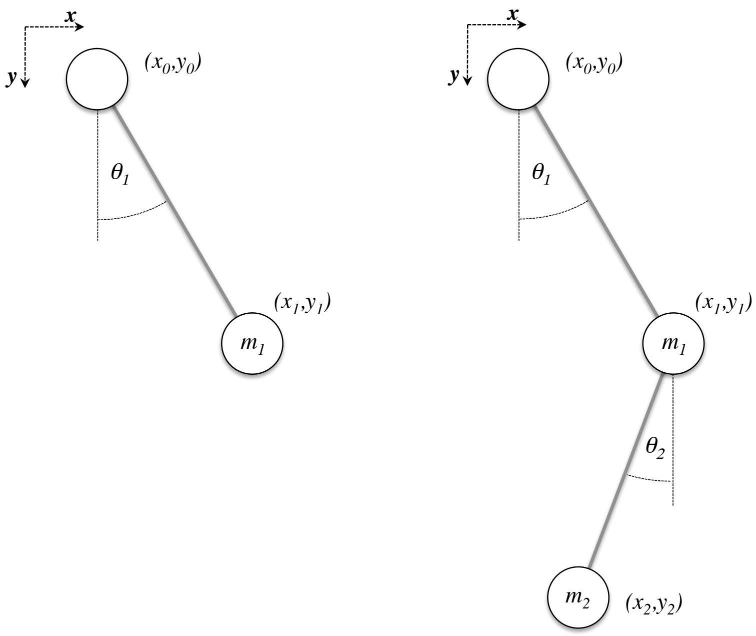

Figure 3: Cartoon of single pendulum suspension (left) and of the double - composite - pendulum (right).

2.2 Forces

All forces in this section are written in the form of $\overrightarrow{F} = \{F_x, F_y\}$ where F_x and F_y are the force components along the x and the y axis respectively. The forces acting on the simple pendulum are therefore:

- Gravity: acts vertically, $\overrightarrow{F_{g_1}} = \{0, m_1 g\}$
- Elasticity of suspension wire (spring-like force): it acts along the direction of the wire and it can be written as $\overrightarrow{F_{k_1}} = -k_1 (d_1 - l_1) \{\sin\theta_1, \cos\theta_1\}$. This force can be expressed also in the form $\overrightarrow{F_{k_1}} = -k_1 \max(d_1 - l_1) \{\sin\theta_1, \cos\theta_1\}$ such that the elastic force will work only when the wire is stretched ($d_1 > l_1$) and not while it is compressed (this option can be selected by the user in the "Augmented Reality Pendulum" sketch).
- Anelasticity of suspension wire (velocity damping-like force): acts along the direction of the wire, $\overrightarrow{F_{\delta k_1}} = -\delta k_1 \dot{d_1}$ where $\dot{d_1} = \frac{1}{d_1} [(x_1 - x_0(t))(\dot{x}_1 - \dot{x}_0) + (y_1 - y_0)(\dot{y}_1 - \dot{y}_0)] \{\sin\theta_1, \cos\theta_1\}$. Also this force should take into account of extension/compression of the wire as for the above elastic force
- Gas damping (velocity damping): assuming small velocity regime i.e. damping linearly proportional to velocity, it can be written as $\overrightarrow{F_{v_1}} = -\beta\{v_{1_x}, v_{1_y}\} = -\beta\{\dot{x}_1, \dot{y}_1\}$. Note that it is directed in opposite direction with respect to the instantaneous velocity.

Diagrams of all the above forces are shown in Fig. 4-left.

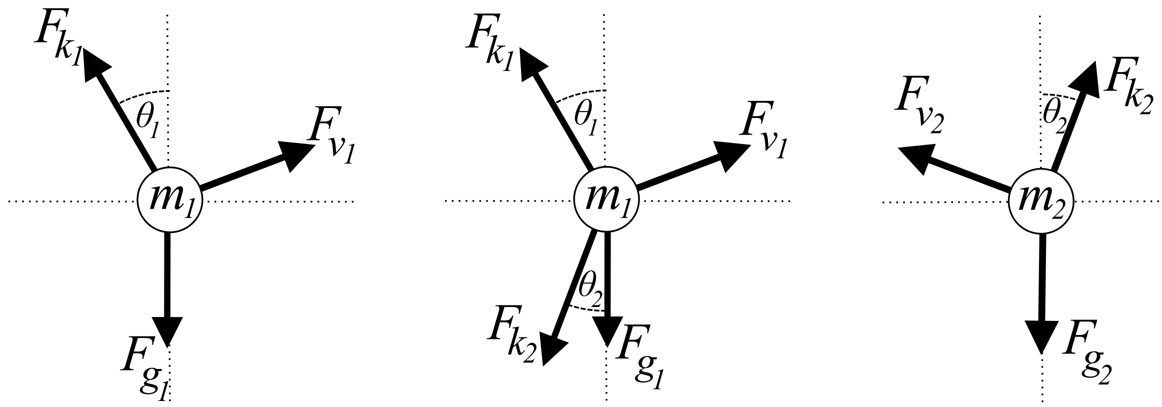

Figure 4: Single pendulum case: diagram of forces acting on mass m_1 (left). Double pendulum: diagram of forces acting on m_1 (center) and mass m_2 (right).

2.3 Equations of motion

Simple pendulum equation of motions can be therefore written as

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1 (d_1 - l_1) \sin\theta_1 - \beta\dot{x}_1 \\ &\quad -\frac{\delta k_1}{d_1} [(x_1 - x_0(t))(\dot{x}_1 - \dot{x}_0) + (y_1 - y_0)(\dot{y}_1 - \dot{y}_0)] \sin\theta_1 \qquad (1) \\ m_1\ddot{y}_1 &= m_1 g - k_1 (d_1 - l_1) \cos\theta_1 - \beta\dot{y}_1 \\ &\quad -\frac{\delta k_1}{d_1} [(x_1 - x_0(t))(\dot{x}_1 - \dot{x}_0) + (y_1 - y_0)(\dot{y}_1 - \dot{y}_0)] \cos\theta_1 \qquad (2) \end{aligned}$$

3 Double composite pendulum

3.1 Constants and definitions

We consider now the case of a composite pendulum system with a second pendulum #2 hanging from pendulum #1, as shown in Fig. 3-right. The relevant physical parameters for pendulum #1 are the same as above and similarly the ones for pendulum #2 are:

- pendulum #2 has coordinates (x_2, y_2) and it hangs from (x_1, y_1)
- the length at rest is l_2, the mass m_2, the elastic constants k_2, the anelasticity coefficient δk_2 and the gas damping β_2.
- $d_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the actual length of the suspension wire.
- θ_2 is the angle that pendulum #2 forms with the vertical axis y and, as above, $\sin\theta_2 = \frac{(x_2 - x_1)}{d_2}$ and $\cos\theta_2 = \frac{(y_2 - y_1)}{d_2}$ to take into account only the extension part of the force.

3.2 Forces

3.2.1 Forces on mass #1

All forces on mass #1 described in the previous sections can be expressed as above. The only additional force on mass #1 comes from the elasticity of pendulum #2 wire which pulls from below. This can be written as

- $\overrightarrow{F_{k_{12}}} = +k\,(d_2 - l_2)\,\{\sin\theta_2, \cos\theta_2\}$ (note the positive sign) and also this force can be expressed as $\overrightarrow{F_{k_{12}}} = +k\max(d_2 - l_2, 0)\,\{\sin\theta_2, \cos\theta_2\}$

The contribution from anelasticity of wire #2 is neglected for the moment. Diagrams of all the above forces are shown in Fig. 4-centre.

3.2.2 Forces on mass #2

The forces acting on pendulum #2 follow the usual expressions seen before:

- Gravity $\overrightarrow{F_{g_2}} = m_2 g\{0, 1\}$
- Elasticity of wire $\overrightarrow{F_{k_2}} = -k_2\,(d_2 - l_2)\,\{\sin\theta_2, \cos\theta_2\}$ (or in the form $F = -k_2\max(d_2 - l_2, 0)$)
- Anelasticity of wire $\overrightarrow{F_{\delta k_2}} = -\delta k \dot{d}_2$ where $\dot{d}_2 = \frac{1}{d_2}\left[(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1)\right[\{\sin\theta_2, \cos\theta_2\}$
- Residual gas damping $\overrightarrow{F_{v_2}} = -\beta\{v_{2_x}, v_{2_y}\} = -\beta\{\dot{x}_2, \dot{y}_2\}$

Diagrams of all the above forces are shown in Fig. 4-right.

3.3 Equations of motion

With the above forces, we can finally write the equations of motion for the composite pendulum system as follows:

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1\,(d_1 - l_1)\sin\theta_1 - \beta_1\dot{x}_1 + k_2\,(d_2 - l_2)\sin\theta_2 \\
&\quad -\delta k_1 \frac{1}{\sqrt{d_1}}\left((x_1 - x_0(t))(\dot{x}_1 - \dot{x}_0) + (y_1 - y_0)(\dot{y}_1 - \dot{y}_0)\right)\sin\theta_1 \qquad (3) \\
m_1\ddot{y}_1 &= -k_1\,(d_1 - l_1)\cos\theta_1 - \beta\dot{y}_1 + k_2\,(d_2 - l_2)\cos\theta_2 - m_1 g \\
&\quad -\delta k_1 \frac{1}{\sqrt{d_1}}\left((x_1 - x_0(t))(\dot{x}_1 - \dot{x}_0) + (y_1 - y_0)(\dot{y}_1 - \dot{y}_0)\right)\cos\theta_1 \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
m_2\ddot{x}_2 &= -k_2\left(d_2-l_2\right)\sin\theta_2-\beta\dot{x}_2 \\
&\quad -\delta k_2\frac{1}{\sqrt{d_2}}\left(\left(x_2-x_1\right)\left(\dot{x}_2-\dot{x}_1\right)+\left(y_2-y_1\right)\left(\dot{y}_2-\dot{y}_1\right)\right)\sin\theta_1 \qquad (5)\\
m_2\ddot{y}_2 &= -k_2\left(d_2-l_2\right)\cos\theta_2-\beta\dot{y}_2-m_2 g \\
&\quad -\delta k_2\frac{1}{\sqrt{d_2}}\left(\left(x_2-x_1\right)\left(\dot{x}_2-\dot{x}_1\right)+\left(y_2-y_1\right)\left(\dot{y}_2-\dot{y}_1\right)\right)\cos\theta_1 \qquad (6)
\end{aligned}
$$

4 Leapfrog Method for integration of equations of motion

For the solution of the equation of motion the "Pendulum" and the "Augmented-Reality Pendulum" sketches make use of the Leapfrog method[1], a well know method for the numerical integration of differential equations, which is commonly used to study of the time evolution of physical systems. With this method positions velocities and accelerations are derived at "interleaved" times (hence "leapfrog"), and this proves to provide reliable results as long as the time interval $\Delta t = (t_{i+1} - t_i)$ between successive samplings i (the "leapfrog" step) is smaller than any characteristic time in the system. In the "Pendulum" and "Augmented-Reality Pendulum" cases this condition can be expressed as

$$\Delta t \ll \frac{1}{2\pi}\sqrt{\frac{g}{d_{1,2}}} \quad \text{and} \quad \Delta t \ll \frac{1}{2\pi}\sqrt{\frac{k_{1,2}}{m_{1,2}}}$$

for the different values of k_1 and k_2, m_1 and m_2, and d_1 and d_2.

In the "Pendulum" and in the "Augmented-Reality Pendulum" sketches, the values of velocities are initially set zero ($t = 0$) and initial values of the coordinates can be arbitrarily chosen. The values of accelerations are derived only at a later stage and therefore can be initially ignored. Then, at any successive times t_i, we perform the following iteration:

1. calculate positions as $x\left(t_{i+1}\right) = x\left(t_i\right) + \dot{x}\left(t_i\right)\left(t_{i+1} - t_i\right)$ (i.e. using the previously calculated value of velocity)
2. calculate accelerations at time t_{i+1} following the equations of motion above, and using the above values of position $\left(x\left(t_{i+1}\right)\right)$ and velocity $\left(\dot{x}\left(t_i\right)\right)$
3. calculate velocities for t_{i+1} as $\dot{x}\left(t_{i+1}\right) = \ddot{x}\left(t_{i+1}\right) + \dot{x}\left(t_i\right)\left(t_{i+1} - t_i\right)$
4. restart from step 1

[1] See for instance `http://en.wikipedia.org/wiki/Leapfrog_integration` and references therein.